TRINITY COLLEGE

IN NOMINE DOMINI

Specialist Mathematics Units 3,4
Test 6 2018

Topic: Simple harmonic motion and statistical sampling

NAME ______________________________

DATE: 6 September **TIME:** 50 minutes **MARKS**: 52

INSTRUCTIONS:

Standard Items: Pens, pencils, eraser or correction fluid, ruler

Special Items: Three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment)

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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1. (7 marks)

A particle moving in a straight line so its displacement x metres from a fixed-point O at time t seconds is given by $x = 2a\sin\left(\frac{nt}{2}\right) + \frac{b}{2}\cos\left(\frac{nt}{2}\right)$.

(a) Show that the particle is moving in simple harmonic motion about the origin. (3 marks)

$$\dot{x} = an\cos\frac{nt}{2} - \frac{bn}{4}\sin\frac{nt}{2}$$

$$\ddot{x} = -\frac{an^2}{2}\sin\frac{nt}{2} - \frac{bn^2}{8}\cos\frac{nt}{2}$$

$$= -\frac{n^2}{4}\left(2a\sin\frac{nt}{2} + \frac{b}{2}\cos\frac{nt}{2}\right)$$

$$= -\frac{n^2x}{4}$$

(b) Given the particle is initially 20 m to the left of O, has a velocity of 360 ms^{-1} and has an acceleration of 180 ms^{-2}, determine the values of the constants a, b and n. (n > 0). (4 marks)

$-20 = \frac{b}{2}$

$360 = an$

$180 = 5n^2$

$n = 6$ $(n > 0)$

$a = 60$

$b = -40$

2. (10 marks)

A steam driven piston has a displacement, x cm, given by $x(t) = 2\cos 2t - \sqrt{12}\sin 2t$, t seconds after the motion begins.

(a) (i) Show that the displacement equation can be written in the form $x(t) = A\cos(2t + \alpha)$, stating the values of A and α (4 marks)

$2\cos 2t - \sqrt{12}\sin 2t$

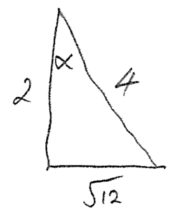
$= 4\left(\frac{2}{4}\cos 2t - \frac{\sqrt{12}}{4}\sin 2t\right)$

$= 4(\cos 2t\cos\alpha - \sin 2t \sin\alpha)$

$= 4\cos(2t + \alpha)$

$= 4\cos(2t + \frac{\pi}{3})$

$A = 4$



$\cos\alpha = \frac{2}{4}$

$\alpha = 60^0$ or $\frac{\pi}{3}$

(ii) Show that the piston is moving in simple harmonic motion i.e. $a = -k^2x$ stating the value of k. (3 marks)

$\dot{x} = -8\sin(2t + \frac{\pi}{3})$

$\ddot{x} = -16\cos(2t + \frac{\pi}{3})$

$= -2^2x$

$k = 2$

(b) Determine the period of the motion and the first 2 times when it changes direction. (3 marks)

PERIOD $= \frac{2\pi}{k}$

$= \pi$

$-8\sin(2t + \frac{\pi}{3}) = 0$

$2t + \frac{\pi}{3} = \pi$

$t = \frac{\pi}{3}, \frac{5\pi}{6}$

$1.05, 2.62$

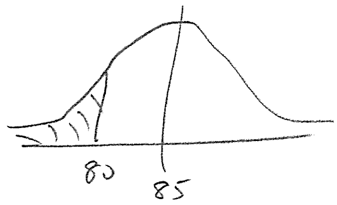
3. (4 marks)

A company produces calculator batteries which are known to have a life expectancy which is normally distributed with a mean of 85 hours and a standard deviation of 8 hours and 12 minutes.

(a) Determine the probability that a randomly selected battery will fail before 80 hours of use. (2 marks)

$X \sim N(85, 8.2^2)$

$P(X < 80) = 0.2710$



The company tested a random sample of 50 batteries to ensure standards were being met. After testing the life of those 50 batteries, the sample was determined to have a mean life of 83.72 hours.

(b) Construct a 90% confidence interval based on this sample (2 marks)

$\bar{x} = 83.72$

$z = 1.645$

$s = 8.2$

$n = 50$

$81.81 \le \mu \le 85.63$

4 (~~3~~ 4 marks)

A particle has an initial velocity of 10 ms^{-1} and is at the origin. The acceleration at any time is a function of its velocity at that time, with $a = (1 + v^2)\ ms^{-2}$
Determine the exact distance the particle moves in increasing its velocity to double its initial velocity.

$a = 1 + v^2$

$v\dfrac{dv}{dx} = 1 + v^2$

$\dfrac{v}{1+v^2}\, dv = dx$

$\frac{1}{2}\ln|1+v^2| = x + c$

$x = 0 \quad v = 10$

$\frac{1}{2}\ln(101) = 0 + c$

$\frac{1}{2}\ln 101 = c$

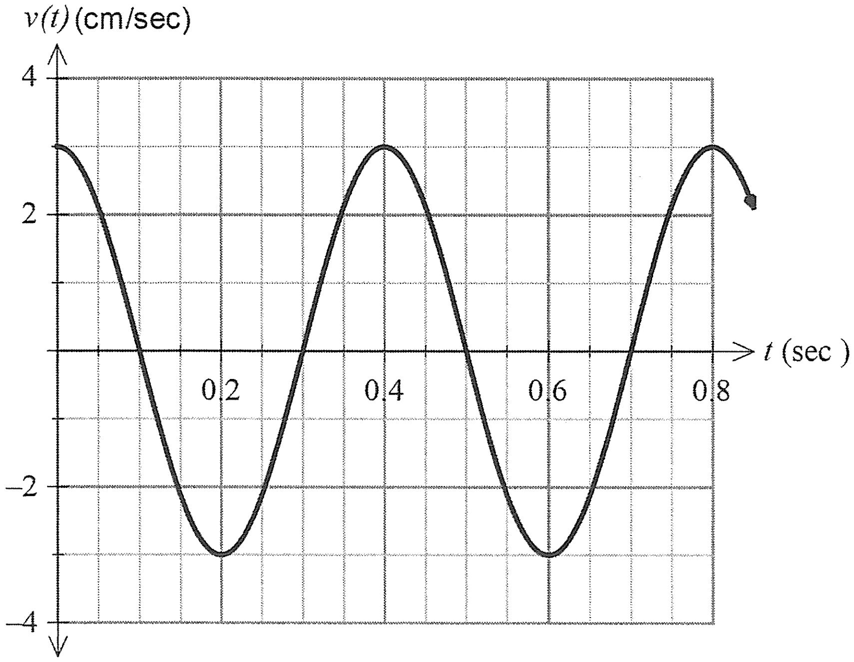
$v = 20$

$\frac{1}{2}\ln 401 = x + \frac{1}{2}\ln 101$

$\frac{1}{2}\ln\frac{401}{101} = x$

5. (10 marks)

A particle performs simple harmonic motion with velocity $v(t)$, measured in cm per second, defined by the graph below.



(a) The velocity $v(t)$ is given by $v(t) = 3\cos(bt)$. Determine the value of b. (2 marks)

PERIOD $= 0.4$

$0.4 = \frac{2\pi}{b}$

$v(t) = 3\cos 5\pi t$

$b = 5\pi$

The displacement of the particle is given by $x(t)$ centimetres, with $x(0) = 1$.

(b) Determine $x(t)$ in the form $x(t) = A\sin(bt) + C$. (2 marks)

$x(t) = \int 3\cos 5\pi t \, dt$

$x(t) = \frac{3}{5\pi} \sin 5\pi t + c$

$t = 0$
$x = 1$

$1 = 0 + c$

$x(t) = \frac{3}{5\pi} \sin 5\pi t + 1$

(c) Calculate how far the particle will travel in the first second.

(3 marks)

$$\int_0^1 |v(t)|\,dt = \frac{6}{\pi}$$

$$= 1.91$$

(d) Show that $\dfrac{d^2x}{dt^2} = -k(x-1)$ for an appropriate value of k. (3 marks)

$$x = \frac{3}{5\pi}\sin 5\pi t + 1$$

$$\dot{x} = 3\cos 5\pi t$$

$$\ddot{x} = -15\pi \sin 5\pi t$$

$$= -15\pi \times \frac{5\pi}{3}(x-1)$$

$$= -25\pi^2(x-1)$$

$$x = \frac{3}{5\pi}\sin 5\pi t + 1$$

$$x - 1 = \frac{3}{5\pi}\sin 5\pi t$$

$$\frac{5\pi}{3}(x-1) = \sin 5\pi t$$

6. (7 marks)

On the basis of the results obtained from a random sample of 81 bags produced by a mill, the 95% confidence interval for the mean weight of flour in a bag is found to be 514.56 g < μ < 520.44 g.

(a) Determine the value of $\bar{x}$, the mean weight of the sample. (1 mark)

$$\frac{520.44 + 514.56}{2} = 517.5$$

(b) Determine the value of the standard deviation of the normal population from which the sample is drawn. (2 marks)

$$517.5 - 1.960 \times \frac{s}{\sqrt{81}} = 514.56$$

$$s = 13.5$$

(c) Calculate the 99% confidence interval for the mean weight of flour in a bag. (2 marks)

$\bar{x} = 517.5$

$z = 2.576$

$s = 13.5$

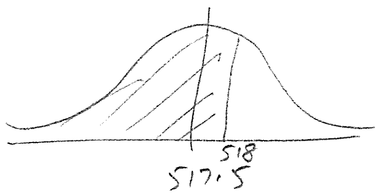
$n = 81$

$$513.64 \le \mu \le 521.36$$

(d) Using the sample mean from part (a) as the best estimate for the population mean, what is the probability that the sample mean of a larger sample of 225 bags is less than 518 g? (2 marks)

$$X \sim N\left(517.5, \left(\frac{13.5}{\sqrt{225}}\right)^2\right)$$

$$P(X < 518) = 0.7107$$



7. (10 marks)

An internet service provider plans to sample the volume of content downloaded per day by customers subscribing to their ADSL20 plan. From recent research, the company knew that the standard deviation of the volume of downloads per customer was 1.4 GB.

(a) Determine how large a sample the company should take in order to be 90% confident that the mean volume of downloads per customer calculated from their sample is within 0.25 GB of the true population mean. (2 marks)

$$0.25 = \frac{1.645 \times 1.4}{\sqrt{n}}$$

$$n = 84.8$$
$$\approx 85$$

(b) A random selection of 25 subscribers was made and the total volume downloaded by these customers over a 24-hour period was 120 GB. Calculate a 95% confidence interval for the mean volume of content downloaded per day by a customer. (4 marks)

$\bar{x} = \frac{120}{25} = 4.8$ $\quad z = 1.960 \quad s = 1.4 \quad n = 25$

$$4.25 \leq \mu \leq 5.35$$

(c) If the company repeated the random sampling process and subsequent 95% confidence interval calculations from part (b) a total of 40 times, how many of the intervals calculated would you expect to contain the true population mean? Justify your answer. (2 marks)

$$0.95 \times 40 = 38$$

(d) A claim is made that the mean usage of data is 5GB. Comment. (2 marks)

FALLS INSIDE 95% CONFIDENCE INTERVAL

95% CONFIDENT MEAN IS IN THIS INTERVAL.